Involutions on surfaces of general type with $p_g = 0$

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3. Involutions on Campedelli surfaces [Calabri, Mendes Lopes, Pardini (2008)]
   They consider birational types and branch divisors of quotients by involutions.

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- $\sim$: the birationality between surfaces

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  [Calabri, Ciliberto, Mendes Lopes (2007)]
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So we exclude these two cases.

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- $d = 2, 4$ if $K_S^2 = 3, 4, 5, 6$
- $d = 2$ if $K_S^2 = 7, 8$

  [Mendes Lopes, Pardini (2007)]

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Theorem (Shin)

Let S be a minimal surface of general type with $p_g = 0$ having an involution σ. Assume that the bicanonical map φ is composed with σ. Then the quotient S/σ is rational for $K_S^2 = 5, 6, 7, 8$.

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Examples for each K_S^2 and d

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K_S^2	d	W	Z	Examples
2	8	rational	rational	[Campedelli (1932)], [Burniat (1966)], [Kulikov (2004)]
	8	$\sim$ Enriques surface	rational	[Kulikov (2004)]
3	2	rational	rational	[Rito (2010)]
	2	$\sim$ Enriques surface	$\sim$ Enriques surface	[Mendes Lopes, Pardini (2004)]
	4	rational	rational	[Burniat (1966)]
	4	$\sim$ Enriques surface	rational	[Keum (1988)], [Naie (1994)]

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6	2	rational	rational	[Inoue (1994)], [Mendes Lopes, Pardini (2004)], [Rito (2011)]
	4	rational	rational	[Burniat (1966)]
7	2	rational	rational	[Inoue (1994)], [Mendes Lopes, Pardini (2001)], [Rito (2011)]
8	2	rational	rational	[Mendes Lopes, Pardini (2001)], [Pardini (2003)]

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- $k :=$ the number of isolated fixed points by σ on S
- $B_0 := \tilde{\pi}(\epsilon^*(R))$, where R is a fixed divisor of σ on S
- $\Gamma_{(m,n)} :=$ m is $p_a(\Gamma)$ and n is the self intersection number of Γ

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7	0	$\overset{\Gamma_0}{(2,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	minimal properly elliptic, or of general type whose the minimal model has $K^2 = 1$
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(3,-2)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$, and if W is birational to an Enriques surface then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ or $\overset{\Gamma_0}{(3,-2)}$.

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Theorem (Lee, Shin)

Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 7$ having an involution σ. If W is birational to an Enriques surface then $k = 9$, $K_W^2 = -2$, and the branch divisor $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ or $\underset{(3,-2)}{\Gamma_0}$.

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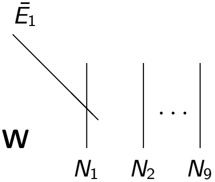
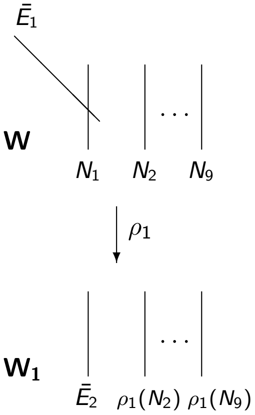
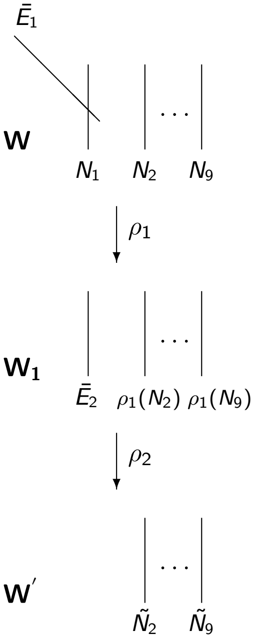
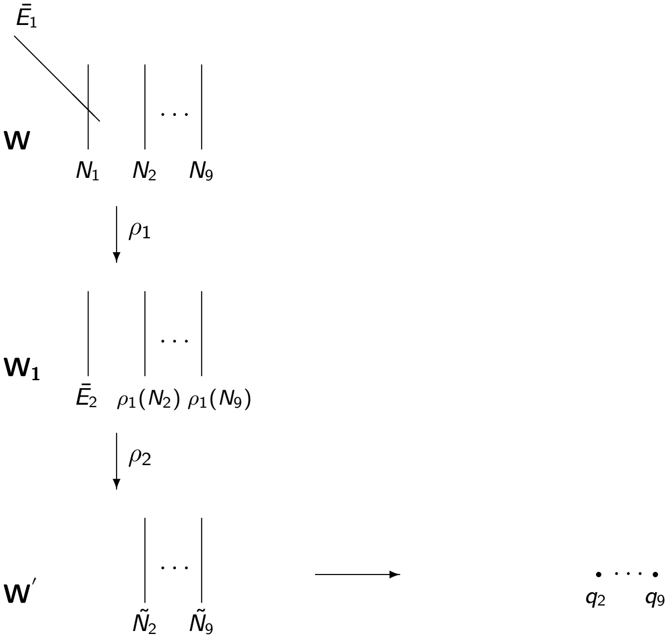
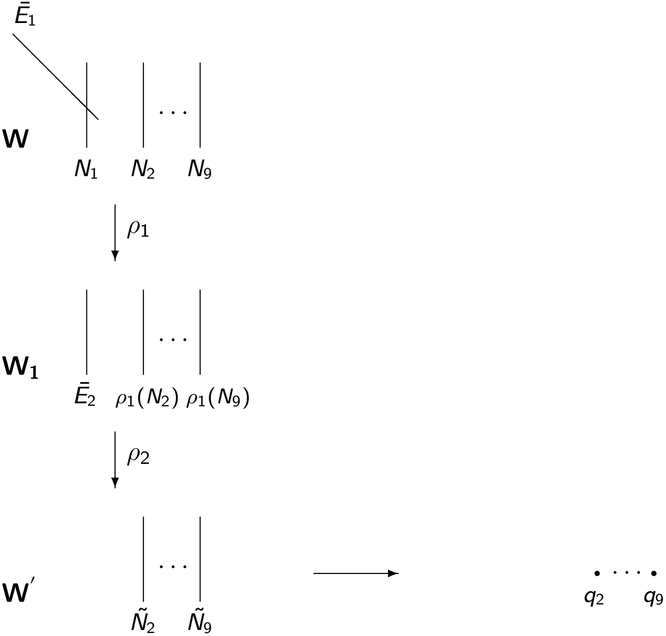
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(3,-2)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$, and if W is birational to an Enriques surface then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ or $\overset{\Gamma_0}{(3,-2)}$.


$\bar{E}_1$
W
N_1
N_2
$\dots$
N_9

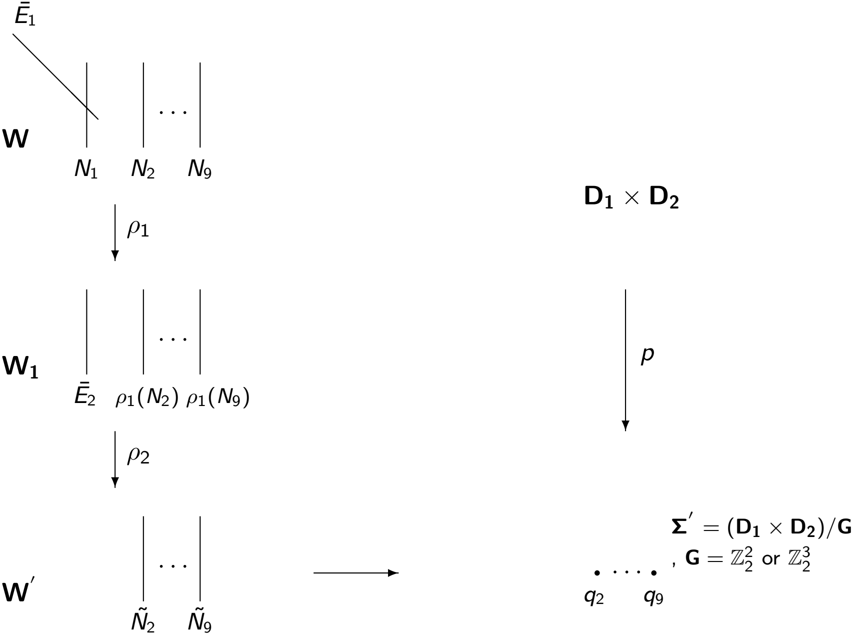

$\bar{E}_1$
$\cdots$
W
N_1 N_2 N_9
ρ_1
$\cdots$
W_1
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$

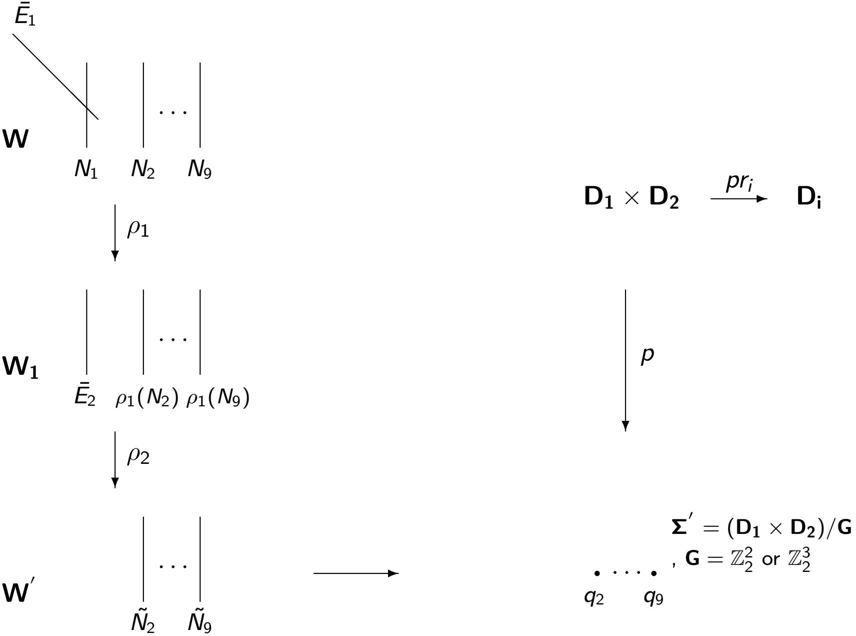

$\bar{E}_1$
W
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}'$
$\tilde{N}_2$ $\tilde{N}_9$

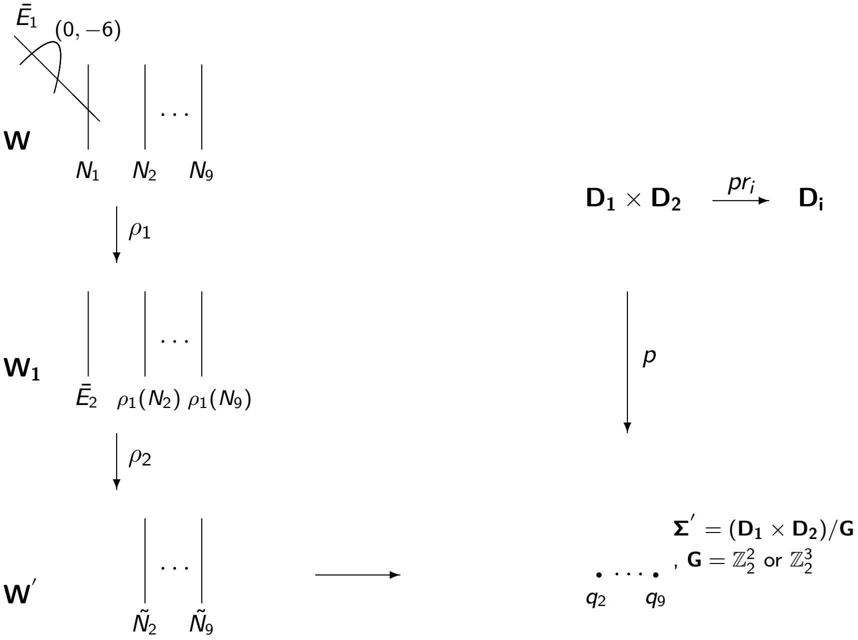

$\bar{E}_1$
W
N_1 N_2 N_9
ρ_1
W_1
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
W'
$\tilde{N}_2$ $\tilde{N}_9$
q_2 q_9



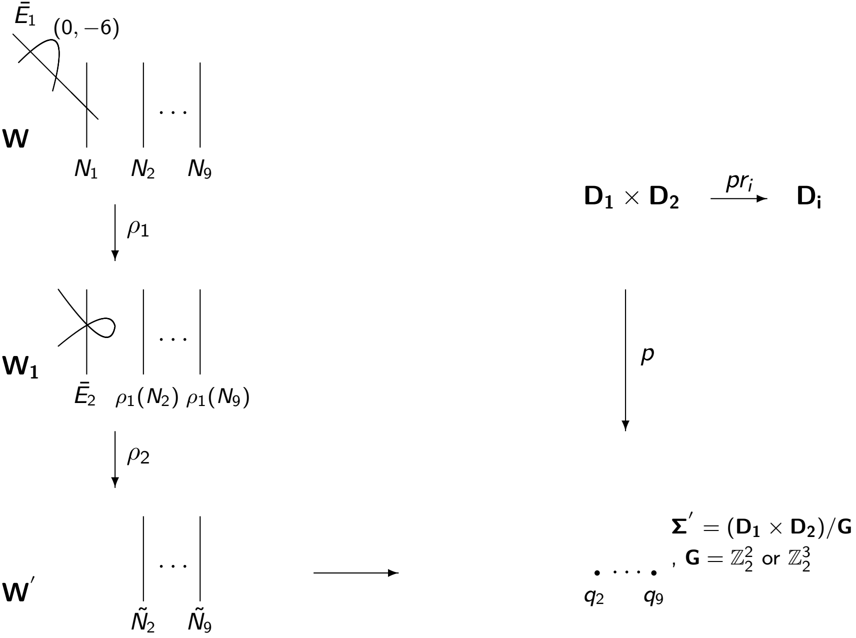
$\mathbf{\Sigma}' = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$, $\mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$

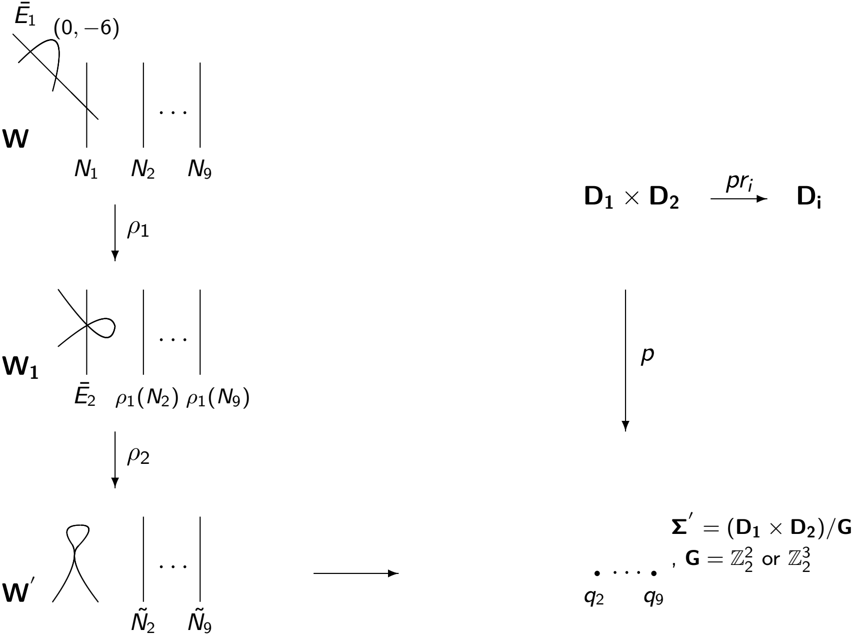

$\bar{E}_1$
W
N_1 N_2 N_9
ρ_1
W$_1$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
W$'$
$\tilde{N}_2$ $\tilde{N}_9$
$\mathbf{D_1} \times \mathbf{D_2}$
p
$\mathbf{\Sigma}' = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
, $\mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$
q_2 q_9


$\bar{E}_1$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}'$
$\tilde{N}_2$ $\tilde{N}_9$
$\mathbf{D_1} \times \mathbf{D_2} \xrightarrow{pr_i} \mathbf{D_i}$
p
q_2 q_9
$\mathbf{\Sigma}' = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
, $\mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$

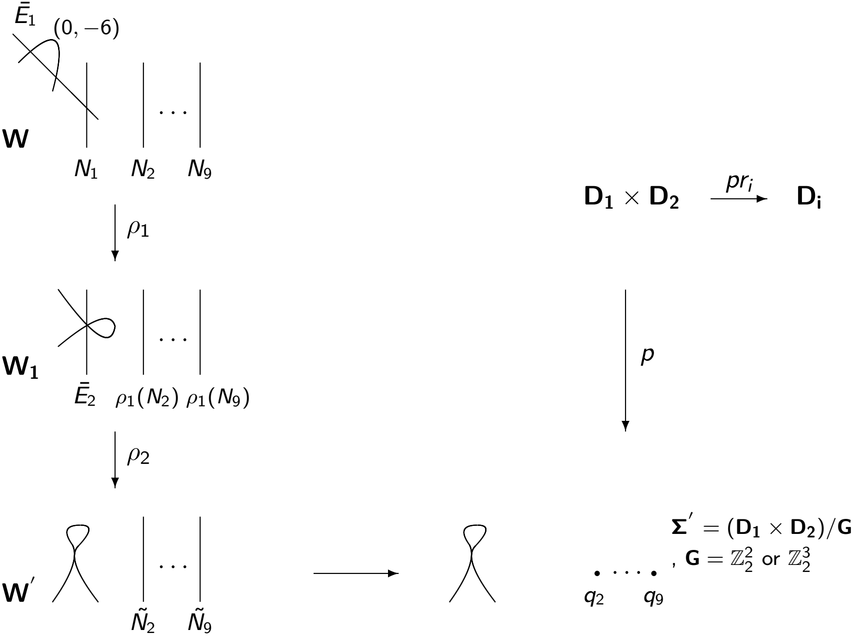


$\bar{E}_1$
$(0,-6)$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}'$
$\tilde{N}_2$ $\tilde{N}_9$
$\mathbf{D_1} \times \mathbf{D_2} \xrightarrow{pr_i} \mathbf{D_i}$
p
$\mathbf{\Sigma}' = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
$, \mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$
q_2 q_9

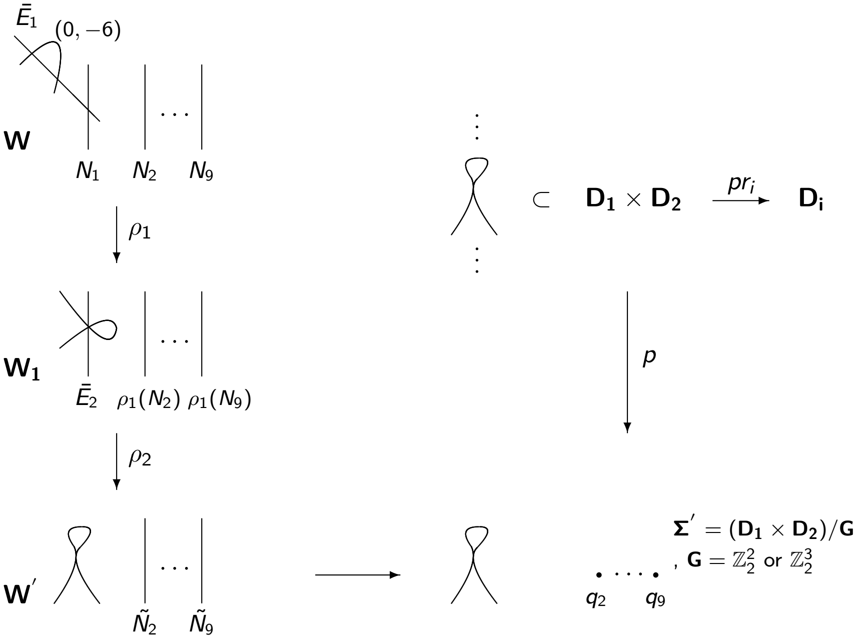

$\bar{E}_1$
$(0, -6)$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}^{'}$
$\tilde{N}_2$ $\tilde{N}_9$
$\mathbf{D_1} \times \mathbf{D_2} \xrightarrow{pr_i} \mathbf{D_i}$
p
q_2 q_9
$\mathbf{\Sigma}^{'} = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
$, \mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$

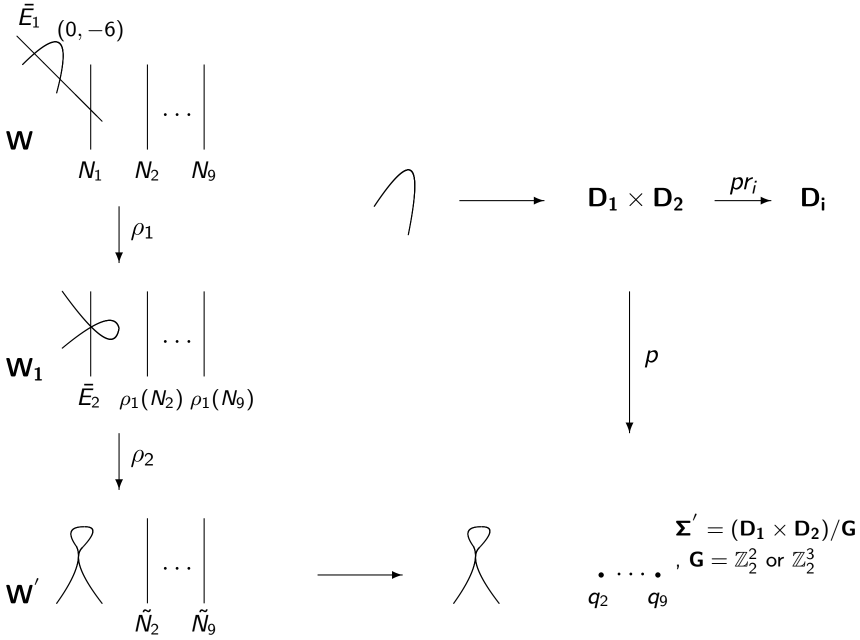


$\bar{E}_1$
(0, −6)
W
N_1
N_2
N_9
ρ_1
W_1
$\bar{E}_2$
$\rho_1(N_2)$
$\rho_1(N_9)$
ρ_2
W'
$\tilde{N}_2$
$\tilde{N}_9$
$D_1 \times D_2$
pr_i
D_i
p
$\Sigma' = (D_1 \times D_2)/G$
, $G = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$
q_2
q_9



$\bar{E}_1$
(0, −6)
W
N_1 N_2 N_9
ρ_1
W_1
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
W'
$\tilde{N}_2$ $\tilde{N}_9$
$D_1 \times D_2$ $\xrightarrow{pr_i}$ D_i
p
q_2 q_9
$\Sigma' = (D_1 \times D_2)/G$
, $G = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$


$\bar{E}_1$
$(0,-6)$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}'$
$\tilde{N}_2$ $\tilde{N}_9$
$\subset \quad \mathbf{D_1} \times \mathbf{D_2} \xrightarrow{pr_i} \mathbf{D_i}$
p
q_2 q_9
$\mathbf{\Sigma}' = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
$, \ \mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$


$\bar{E}_1$
$(0,-6)$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}^{'}$
$\tilde{N}_2$ $\tilde{N}_9$
$\mathbf{D_1} \times \mathbf{D_2}$
pr_i
$\mathbf{D_i}$
p
q_2 q_9
$\mathbf{\Sigma}^{'} = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
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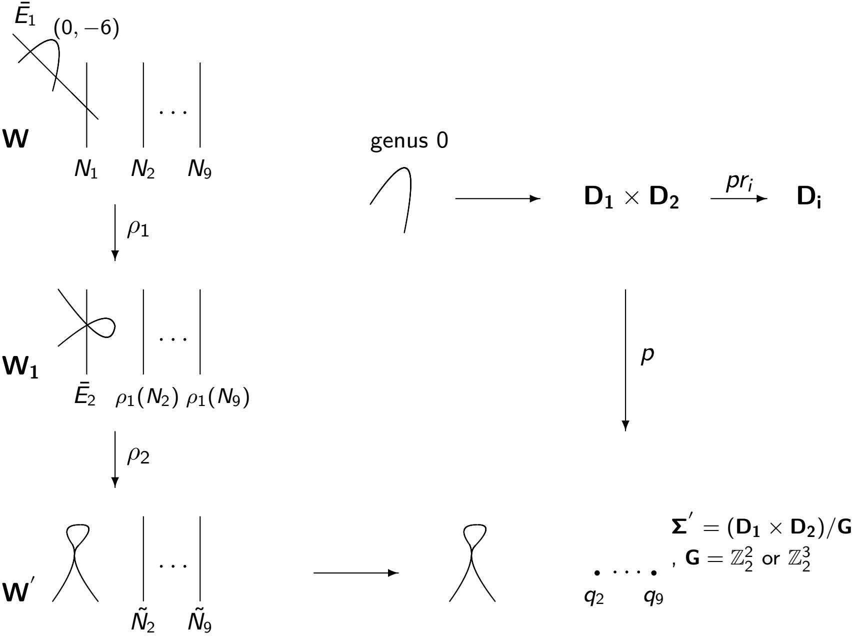
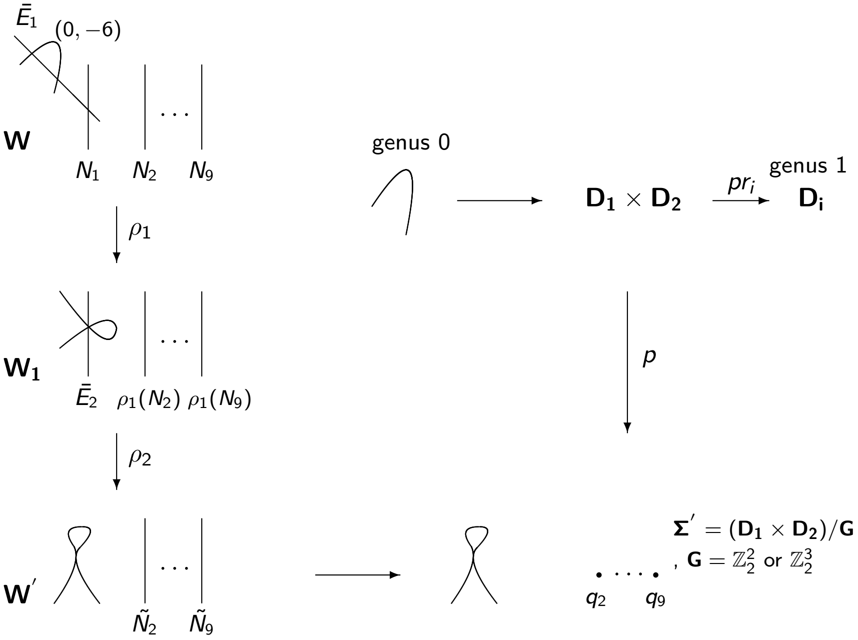
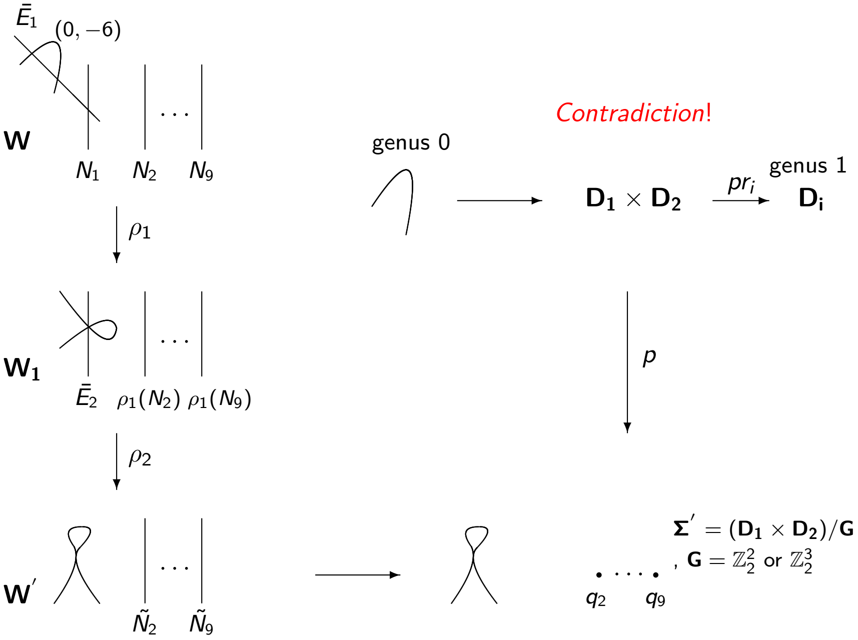

$\bar{E}_1$
$(0,-6)$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}'$
$\tilde{N}_2$ $\tilde{N}_9$
genus 0
$\mathbf{D_1 \times D_2}$
pr_i
$\mathbf{D_i}$
p
q_2 q_9
$\mathbf{\Sigma}' = (\mathbf{D_1 \times D_2})/\mathbf{G}$
$, \mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$


$\bar{E}_1$
$(0, -6)$
$\mathbf{W}$
N_1 N_2 N_9
ρ_1
$\mathbf{W_1}$
$\bar{E}_2$ $\rho_1(N_2)$ $\rho_1(N_9)$
ρ_2
$\mathbf{W}^{'}$
$\tilde{N}_2$ $\tilde{N}_9$
genus 0
$\mathbf{D_1 \times D_2}$
pr_i
genus 1
$\mathbf{D_i}$
p
$\mathbf{\Sigma}^{'} = (\mathbf{D_1 \times D_2})/\mathbf{G}$
, $\mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$
q_2 q_9



$\bar{E}_1$
$(0, -6)$
W
N_1
N_2
N_9
ρ_1
Contradiction!
genus 0
genus 1
$\mathbf{D_1 \times D_2}$
pr_i
$\mathbf{D_i}$
$\mathbf{W_1}$
$\bar{E}_2$
$\rho_1(N_2)$
$\rho_1(N_9)$
p
ρ_2
$\mathbf{\Sigma}' = (\mathbf{D_1} \times \mathbf{D_2})/\mathbf{G}$
$, \mathbf{G} = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$
$\mathbf{W}'$
q_2
q_9
$\tilde{N}_2$
$\tilde{N}_9$

Birational types and branch divisors of the quotient of a minimal surface of general type with $p_g = 0$ and $K^2 = 7$ (cf. [Lee, Shin (2010)])

k	K_W^2	B_0	W
5	2	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
7	1	$\overset{\Gamma_0}{(3,2)}$	minimal of general type
7	0	$\overset{\Gamma_0}{(2,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	minimal properly elliptic, or of general type whose the minimal model has $K^2 = 1$
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(3,-2)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$, and if W is birational to an Enriques surface then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ or $\overset{\Gamma_0}{(3,-2)}$.

Examples for W birational to an Enriques surface

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1. Example 4.1 in The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geq 7$ [Mendes Lopes, Pardini (2001)]
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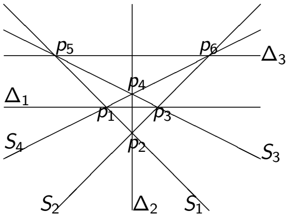

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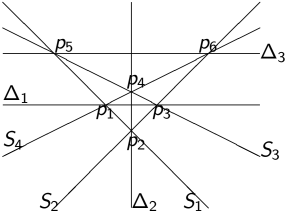

$D_1 := \Delta_1 + f_2 + S_1 + S_2$, $D_2 := \Delta_2 + f_3$,
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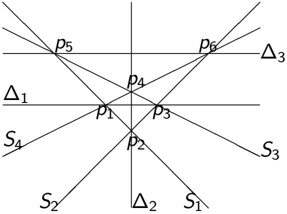

$$D_1 := \Delta_1 + f_2 + S_1 + S_2,\ D_2 := \Delta_2 + f_3,$$
$$D_3 := \Delta_3 + f_1 + f_1' + S_3 + S_4,$$
$$L_1 := 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6,$$
$$L_2 := 6l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6,$$
$$L_3 := 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6$$

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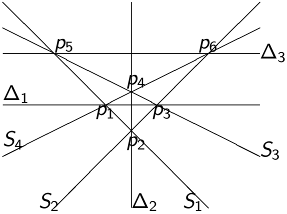

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$\Rightarrow 2L_1 \equiv D_2 + D_3,$

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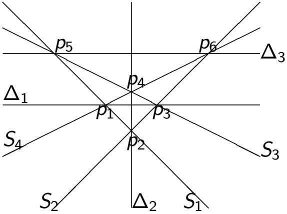

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$\Rightarrow 2L_1 \equiv D_2 + D_3$, $2L_2 \equiv D_1 + D_3$, $2L_3 \equiv D_1 + D_2$.

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$\Rightarrow$ A minimal surface S of general type with $p_g(S) = 0$ and $K_S^2 = 7$ having involutions γ_1, γ_2 and γ_3 induced by a bidouble cover. (i.e. $\mathbb{Z}_2^2$-cover)

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$\Rightarrow$ We obtain the following table:

	k	$K_{W_i}^2$	B_0	W_i
(S, γ_1)	11	-4	$\underset{(3,0)}{\Gamma_0} + \underset{(2,-2)}{\Gamma_1}$	rational
(S, γ_2)	9	-2	$\underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$	birational to an Enriques surface
(S, γ_3)	9	-2	$\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$	rational

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Examples for W birational to an Enriques surface

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(S, γ_3)	7	0	$\underset{(2,-2)}{\Gamma_0}$	minimal properly elliptic

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(S, γ_2)	9	-2	$\underset{(3,-2)}{\Gamma_0}$	birational to an Enriques surface
(S, γ_3)	7	0	$\underset{(2,-2)}{\Gamma_0}$	minimal properly elliptic

Birational types and branch divisors of the quotient of a minimal surface of general type with $p_g = 0$ and $K^2 = 7$ (cf. [Lee, Shin (2010)])

k	K_W^2	B_0	W
5	2	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
7	1	$\overset{\Gamma_0}{(3,2)}$	minimal of general type
7	0	$\overset{\Gamma_0}{(2,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	minimal properly elliptic, or of general type whose the minimal model has $K^2 = 1$
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(3,-2)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$, and if W is birational to an Enriques surface then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ or $\overset{\Gamma_0}{(3,-2)}$.

$K^2 = 2$ [Calabri, Mendes Lopes, Pardini(2008)]

k	K_W^2	B_0	W
4	1	$\emptyset$	minimal of general type
4	0	$\overset{\Gamma_0}{(0,-4)}$	minimal properly elliptic
4	-1	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$
4	-2	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$K^2 = 2$ [Calabri, Mendes Lopes, Pardini(2008)]

k	K_W^2	B_0	W
4	1	$\emptyset$	minimal of general type [Balow (1984), (1985)], [Calabri, Mendes Lopes, Pardini(2008)], [Park, Shin, Urzua (2011)]
4	0	$\overset{\Gamma_0}{(0,-4)}$	minimal properly elliptic [Calabri, Mendes Lopes, Pardini(2008)]
4	-1	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$ W $\sim$Enriques surface [Calabri, Mendes Lopes, Pardini(2008)]
4	-2	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$$K^2 = 3$$

k	K_W^2	B_0	W
5	0	$\overset{\Gamma_0}{(1,-2)}$	minimal properly elliptic
5	-1	$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-6)}$	$\kappa(W) \leq 1$
5	-2	$\overset{\Gamma_0}{(0,-6)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$$K^2 = 3$$

k	K_W^2	B_0	W
5	0	$\overset{\Gamma_0}{(1,-2)}$	minimal properly elliptic [Rito (2012)]
5	-1	$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-6)}$	$\kappa(W) \leq 1$ $W \sim$ Enriques surface [Rito (2012)]
5	-2	$\overset{\Gamma_0}{(0,-6)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$$K^2 = 4$$

k	K_W^2	B_0	W
4	2	$\emptyset$	minimal of general type
4	1	$\overset{\Gamma_0}{(0,-4)}$	minimal of general type or of general type with $K_{W'}^2 = 2$
4	0	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)}$	minimal properly elliptic or of general type with $K_{W'}^2 = 1$ or 2
6	0	$\overset{\Gamma_0}{(2,0)}$	minimal properly elliptic
6	-1	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)}$	$\kappa(W) \leq 1$
6	-2	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-8)}$ $\overset{\Gamma_0}{(0,-6)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$$K^2 = 4$$

k	K_W^2	B_0	W
4	2	$\emptyset$	minimal of general type
4	1	$\overset{\Gamma_0}{(0,-4)}$	minimal of general type or of general type with $K^2_{W'} = 2$
4	0	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)}$	minimal properly elliptic or of general type with $K^2_{W'} = 1$ or 2
6	0	$\overset{\Gamma_0}{(2,0)}$	minimal properly elliptic
6	-1	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)}$	$\kappa(W) \leq 1$ $W \sim$ Enriques surface [Rito (2011)] $\kappa(W) = 1$ [Rito (2011)]
6	-2	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-8)}$ $\overset{\Gamma_0}{(0,-6)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$K^2 = 5$

k	K_W^2	B_0	W
5	1	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
5	0	$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-6)}$	minimal properly elliptic or of general type with $K_{W'}^2 = 1$
7	0	$\overset{\Gamma_0}{(3,2)}$	minimal properly elliptic
7	-1	$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(2,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	$\kappa(W) \leq 1$
7	-2	$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-6)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(1,-4)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$

$K^2 = 5$

k	K_W^2	B_0	W
5	1	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
5	0	$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-6)}$	minimal properly elliptic or of general type with $K_{W'}^2 = 1$
7	0	$\overset{\Gamma_0}{(3,2)}$	minimal properly elliptic
7	-1	$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(2,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	$\kappa(W) \leq 1$ $W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)], $W \sim$ Enriques surface [Rito (2011)] $\kappa(W) = 1$ [Rito (2011)]
7	-2	$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-6)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(1,-4)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$ $W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)] $W \sim$ Enriques surface [Rito (2011)]

$$K^2 = 6$$

k	K_W^2	B_0	W
4	3	$\emptyset$	minimal of general type
4	2	$\overset{\Gamma_0}{(0,-4)}$	minimal of general type or of general type with $K_{W'}^2 = 3$
6	1	$\overset{\Gamma_0}{(2,0)}$	minimal of general type
6	0	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)}$	minimal properly elliptic or of general type with $K_{W'}^2 = 1$

8	0	$\underset{(4,4)}{\Gamma_0}$	minimal properly elliptic
8	-1	$\underset{(4,4)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}$ $\underset{(3,0)}{\Gamma_0}$ $\underset{(3,2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ $\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1}$	$\kappa(W) \leq 1$
8	-2	$\underset{(4,4)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$ $\underset{(3,0)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}$ $\underset{(2,-4)}{\Gamma_0}$ $\underset{(3,2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$ $\underset{(3,2)}{\Gamma_0} + \underset{(0,-6)}{\Gamma_1}$ $\underset{(2,-2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ $\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$ $\underset{(2,0)}{\Gamma_0} + \underset{(1,-4)}{\Gamma_1}$ $\underset{(2,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$	$\kappa(W) \leq 1$

8	0	$\overset{\Gamma_0}{(4,4)}$	minimal properly elliptic
8	-1	$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$
		$\overset{\Gamma_0}{(3,0)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)],
			$W \sim$ Enriques surface [Rito (2011)]
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)}$	$\kappa(W) = 1$ [Rito (2011)]
8	-2	$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$
		$\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(0,-4)}$	
		$\overset{\Gamma_0}{(2,-4)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-6)}$	
		$\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)],
			$W \sim$ Enriques surface [Rito (2011)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(0,-4)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-4)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)]

$K^2 = 7$ (cf. [Lee, Shin (2010)])

k	K_W^2	B_0	W
5	2	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
7	1	$\overset{\Gamma_0}{(3,2)}$	minimal of general type
7	0	$\overset{\Gamma_0}{(2,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	minimal properly elliptic, or of general type whose the minimal model has $K^2 = 1$
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(3,-2)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$ $\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$\kappa(W) \leq 1$, and if W is birational to an Enriques surface then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ or $\overset{\Gamma_0}{(3,-2)}$.

$K^2 = 7$ (cf. [Lee, Shin (2010)])

k	K_W^2	B_0	W
5	2	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
7	1	$\overset{\Gamma_0}{(3,2)}$	minimal of general type
7	0	$\overset{\Gamma_0}{(2,-2)}$	minimal properly elliptic [Chen (2012)],
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	or of general type whose the minimal model has $K^2 = 1$
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$, and
		$\overset{\Gamma_0}{(3,-2)}$	if W is birational to an Enriques surface
		$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ [Inoue (1994)],
		$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$	[Mendes Lopes,Pardini (2001)], [Rito (2011)]
			or $\overset{\Gamma_0}{(3,-2)}$ [Chen (2012)].
		$\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Rito (2009)], [Chen (2012)]
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$	
		$\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Inoue (1994)],[Mendes Lopes,Pardini (2001)], [Rito (2011)]

$$K^2 = 8$$

k	K_W^2	B_0	W
4	4	$\emptyset$	minimal of general type
6	2	$\overset{\Gamma_0}{(2,0)}$	minimal of general type
8	0	 $\overset{\Gamma_0}{(3,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)}$	minimal properly elliptic
10	-2	$\overset{\Gamma_0}{(4,0)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(2,0)}$	rational

$K^2 = 8$

k	K_W^2	B_0	W
4	4	$\emptyset$	minimal of general type [Inoue (1994)], [Mendes Lopes, Pardini (2001)]
6	2	$\overset{\Gamma_0}{(2,0)}$	minimal of general type
8	0		minimal properly elliptic
		$\overset{\Gamma_0}{(3,0)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)}$	[Mendes Lopes, Pardini (2001)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)}$	[Mendes Lopes, Pardini (2001)]
10	-2	$\overset{\Gamma_0}{(4,0)}$	rational
		$\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(2,0)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(2,0)}$	

$K^2 = 2$ [Calabri, Mendes Lopes, Pardini(2008)]

k	K_W^2	B_0	W
4	1	$\emptyset$	minimal of general type [Balow (1984), (1985)], [Calabri, Mendes Lopes, Pardini(2008)], [Park, Shin, Urzua (2011)]
4	0	$\overset{\Gamma_0}{(0,-4)}$	minimal properly elliptic [Calabri, Mendes Lopes, Pardini(2008)]
4	-1	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$ W $\sim$Enriques surface [Calabri, Mendes Lopes, Pardini(2008)]
4	-2	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$$K^2 = 3$$

k	K_W^2	B_0	W
5	0	$\overset{\Gamma_0}{(1,-2)}$	minimal properly elliptic [Rito (2012)]
5	-1	$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-6)}$	$\kappa(W) \leq 1$ $W \sim$ Enriques surface [Rito (2012)]
5	-2	$\overset{\Gamma_0}{(0,-6)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$K^2 = 4$

k	K_W^2	B_0	W
4	2	$\emptyset$	minimal of general type
4	1	$\overset{\Gamma_0}{(0,-4)}$	minimal of general type or of general type with $K_{W'}^2 = 2$
4	0	$\overset{\Gamma_0}{(0,-4)} + \overset{\Gamma_1}{(0,-4)}$	minimal properly elliptic or of general type with $K_{W'}^2 = 1$ or 2
6	0	$\overset{\Gamma_0}{(2,0)}$	minimal properly elliptic
6	-1	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)}$	$\kappa(W) \leq 1$ $W \sim$ Enriques surface [Rito (2011)] $\kappa(W) = 1$ [Rito (2011)]
6	-2	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(0,-8)}$ $\overset{\Gamma_0}{(0,-6)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$

$K^2 = 5$

k	K_W^2	B_0	W
5	1	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
5	0	$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(0,-4)}$	minimal properly elliptic
		$\overset{\Gamma_0}{(0,-6)}$	or of general type with $K_{W'}^2 = 1$
7	0	$\overset{\Gamma_0}{(3,2)}$	minimal properly elliptic
7	-1	$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$
		$\overset{\Gamma_0}{(2,-2)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)],
			$W \sim$ Enriques surface [Rito (2011)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	$\kappa(W) = 1$ [Rito (2011)]
7	-2	$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$
		$\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(0,-4)}$	
		$\overset{\Gamma_0}{(1,-6)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-6)}$	
		$\overset{\Gamma_0}{(1,-4)} + \overset{\Gamma_1}{(1,-2)}$	$W \sim$ Enriques surface [Rito (2011)]
		$\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	

$$K^2 = 6$$

k	K_W^2	B_0	W
4	3	$\emptyset$	minimal of general type
4	2	$\overset{\Gamma_0}{(0,-4)}$	minimal of general type or of general type with $K_{W'}^2 = 3$
6	1	$\overset{\Gamma_0}{(2,0)}$	minimal of general type
6	0	$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(0,-4)}$ $\overset{\Gamma_0}{(1,-4)}$ $\overset{\Gamma_0}{(1,-2)} + \overset{\Gamma_1}{(1,-2)}$	minimal properly elliptic or of general type with $K_{W'}^2 = 1$

8	0	$\overset{\Gamma_0}{(4,4)}$	minimal properly elliptic
8	-1	$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$
		$\overset{\Gamma_0}{(3,0)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)],
			$W \sim$ Enriques surface [Rito (2011)]
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)}$	$\kappa(W) = 1$ [Rito (2011)]
8	-2	$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-4)} + \overset{\Gamma_2}{(0,-4)}$	$\kappa(W) \leq 1$
		$\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(0,-4)}$	
		$\overset{\Gamma_0}{(2,-4)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(0,-6)}$	
		$\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)],
			$W \sim$ Enriques surface [Rito (2011)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(0,-4)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)]
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-4)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Mendes Lopes, Pardini (2004)]

$K^2 = 7$ (cf. [Lee, Shin (2010)])

k	K_W^2	B_0	W
5	2	$\overset{\Gamma_0}{(1,-2)}$	minimal of general type
7	1	$\overset{\Gamma_0}{(3,2)}$	minimal of general type
7	0	$\overset{\Gamma_0}{(2,-2)}$	minimal properly elliptic [Chen (2012)],
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(1,-2)}$	or of general type whose the minimal model has $K^2 = 1$
9	-2	$\overset{\Gamma_0}{(4,2)} + \overset{\Gamma_1}{(0,-4)}$	$\kappa(W) \leq 1$, and
		$\overset{\Gamma_0}{(3,-2)}$	if W is birational to an Enriques surface
		$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(0,-4)}$	then $B_0 = \overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$ [Inoue (1994)],
		$\overset{\Gamma_0}{(4,4)} + \overset{\Gamma_1}{(0,-6)}$	[Mendes Lopes,Pardini (2001)], [Rito (2011)]
			or $\overset{\Gamma_0}{(3,-2)}$ [Chen (2012)].
		$\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Rito (2009)], [Chen (2012)]
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-4)}$	
		$\overset{\Gamma_0}{(2,-2)} + \overset{\Gamma_1}{(2,0)}$	
		$\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)} + \overset{\Gamma_2}{(1,-2)}$	
		$\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(1,-2)}$	$W \sim \mathbb{P}^2$ [Inoue (1994)],[Mendes Lopes,Pardini (2001)],
			[Rito (2011)]

$K^2 = 8$

k	K_W^2	B_0	W
4	4	$\emptyset$	minimal of general type [Inoue (1994)], [Mendes Lopes, Pardini (2001)]
6	2	$\overset{\Gamma_0}{(2,0)}$	minimal of general type
8	0	$\overset{\Gamma_0}{(3,0)}$ $\overset{\Gamma_0}{(3,2)} + \overset{\Gamma_1}{(1,-2)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)}$	minimal properly elliptic [Mendes Lopes, Pardini (2001)] [Mendes Lopes, Pardini (2001)]
10	-2	$\overset{\Gamma_0}{(4,0)}$ $\overset{\Gamma_0}{(3,0)} + \overset{\Gamma_1}{(2,0)}$ $\overset{\Gamma_0}{(2,0)} + \overset{\Gamma_1}{(2,0)} + \overset{\Gamma_2}{(2,0)}$	rational

Thank you for your attention!